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# Global Nonlinear Aerodynamic Modeling Using Multivariate Orthogonal Functions

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**A technique was developed for global modeling of nonlinear aerodynamic coefficients using multivariate orthogonal functions generated from the data. The orthogonality of the modeling functions allowed straightforward determination of an adequate model structure and the associated parameter values. A minimum predicted squared-error criterion was used to determine which orthogonal functions should be retained in the model. Each retained orthogonal function was then decomposed into an expansion of ordinary polynomials in the independent variables so that the final model could be interpreted as selectively retained terms from a multivariable power series expansion. The approach was demonstrated on the Z body axis aerodynamic force coefficient $C_Z$ for the F-18 High Angle of Attack Research Vehicle. The data came from a tabular wind-tunnel data base and covered the entire subsonic flight envelope. For a realistic prediction case, the analytical model predicted $C_Z$ within 5% of the wind-tunnel values. The modeling technique developed in this article was shown to be capable of generating a compact, global analytical representation of nonlinear aerodynamics. The resultant polynomial model demonstrated good predictive capability, global validity, analytical differentiability, and improved insight into the character of the nonlinear aerodynamics.**

## Nomenclature

| | | |
|---|---|---|
| $\boldsymbol{a}$ | = | orthogonal function parameter vector |
| $a_j$ | = | $j$th orthogonal function parameter |
| $b_{ki}$ | = | $i$th parameter for the $k$th orthogonal function |
| $C_Z$ | = | $Z$ body axis aerodynamic force coefficient |
| $c$ | = | wing reference chord |
| $c_i$ | = | $i$th polynomial function parameter |
| $\boldsymbol{e}$ | = | $N$-dimensional modeling error vector |
| $J$ | = | cost function |
| $k$, $\bar{k}$ | = | indices for multivariate orthogonal functions |
| $k_i$ | = | order of the $i$th independent variable |
| $M$ | = | Mach number |
| $m$ | = | number of independent variables |
| $N$ | = | number of sample times |
| $n$ | = | number of retained orthogonal functions |
| $n_{max}$ | = | total number of generated orthogonal functions |
| $n_p$ | = | number of retained polynomial functions |
| $n_t$ | = | number of terms in a candidate model |
| $P$ | = | $N \times n$ matrix of orthogonal functions |
| $\boldsymbol{p}_j$ | = | $j$th $N$-dimensional vector multivariate orthogonal function |
| $q$ | = | body axis pitch rate |
| $V$ | = | airspeed |
| $\boldsymbol{w}_k$ | = | $k$th $N$-dimensional vector multivariate polynomial function |
| $X$ | = | $N \times m$ matrix of independent variable vectors |
| $\boldsymbol{x}_i$ | = | $N$-dimensional vector of independent variable $i$ |
| $\bar{y}$ | = | –average value of $y_i$ |
| $\boldsymbol{y}$ | = | $N$-dimensional dependent variable vector |
| $y_i$ | = | $i$th value of the dependent variable |
| $\boldsymbol{1}$ | = | $N$-dimensional vector of ones |
| $\alpha$ | = | angle of attack |
| $\delta_f$ | = | trailing-edge flap deflection |
| $\delta_n$ | = | leading-edge flap deflection |
| $\delta_s$ | = | horizontal stabilator deflection |
| $\gamma_k^j$ | = | orthogonalization scalar |
| $\sigma_p^2$ | = | a priori upper-bound estimate of prediction mean square error |
| $\sigma_0^2$ | = | maximum prediction mean square error |
| $\varphi(k)$ | = | order of multivariate function $k$ |
| $\otimes$ | = | vector element-by-element product |
| $\Leftrightarrow$ | = | implies and is implied by |

*Superscripts*

| | | |
|---|---|---|
| $T$ | = | transpose |
| $-1$ | = | matrix inverse |
| $\cdot$ | = | time derivative |
| $\wedge$ | = | estimate |

## Introduction

MODERN high-performance aircraft operate in flight regimes characterized by nonlinear aerodynamics. Typically, this involves wide variations in angle of attack, sideslip angle, and body axis rotational rates. Activities such as control system design, dynamic analysis, and flight simulation require that the nonlinear aerodynamics be modeled with high fidelity.

An important aspect of accurately modeling nonlinear aerodynamics is determining the mathematical form that relates control surface deflections and aircraft response variables to aerodynamic forces and moments acting on the aircraft, described in terms of nondimensional coefficients. In general, the goal is to find a compact model that still has adequate complexity to capture the nonlinearities. Keeping the number of terms in the model low improves parameter identifiability, resulting in more accurate parameter estimates and good prediction performance.

Models can be loosely classified as local or global. Local models are identified using data from a relatively small region of the independent variable space. Typically, local aerodynamic models are valid for small ranges in angle of attack and sideslip angle at a fixed Mach number. A global model results when the range of validity for the identified model covers a large portion of the flight envelope, e.g., subsonic flight for large ranges in angle of attack and sideslip angle. Given this latter type of (generally nonlinear) model for the aerodynamic coefficients, operations can often be streamlined by replacing many local models with a single global model.

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If the global analytical model for an aerodynamic force or moment coefficient resembles a truncated multivariable power series expansion, some insight can be gained into the nature of the nonlinear aerodynamics. Such insight is not available in a tabular aerodynamic data base where typically linear stability and control derivatives are tabulated as functions of Mach number, angle of attack, and/or sideslip angle. An analytical description of the aerodynamic coefficients in terms of ordinary polynomials would also have a smooth gradient that could be easily computed. Smooth gradients are required in control system design, optimization, global nonlinear stability and control analysis, and in producing local linear models by evaluating partial derivatives at various trim conditions. Global polynomial models could also be updated more easily based on additional data from flight tests, since the changes could be assigned to specific terms of the polynomial function, rather than having to decide how to distribute the changes over the values in the aerodynamic tables.

An important property of any model is predictive capability. The main requirements for good predictive capability are a data set that spans the complete independent variable space and an adequate model structure with a minimum number of model parameters, i.e., a compact model. For aircraft aerodynamic forces and moments, wind-tunnel data generally spans the independent variable space quite well, and thus can be used to determine adequate model structure for a global model with good predictive capability. The connection between model parameterization and predictive capability arises from the fact that overparameterization in the model "fits the noise" to some extent and leads to inaccurate parameter estimates with high variances. This has a detrimental effect on predicting the dependent variable, given new data for the independent variables.

Several problems are encountered when attempting to determine a compact global nonlinear model comprised of polynomial terms in the independent variables using traditional methods. First, it is never clear what the structure of the model should be, or, equivalently, what terms should be retained in the truncated multivariable power series. Several researchers have used concepts from statistical hypothesis testing to select terms for the model from a pool of postulated terms.[1-4] This approach is generally known as stepwise regression. Even if a candidate model structure is adequate, however, the regressors in the resulting least-squares problem can have high correlation (e.g., linear and cubic terms in the same independent variable are often highly correlated). This renders the parameter estimation problem ill-conditioned, causing inaccurate model parameter estimates. Complicating matters further is the fact that the model structure determination and the parameter estimation are coupled, so that adequate model structure is an important prerequisite to accurate parameter estimates. At the same time, evaluating the adequacy of a postulated model structure requires estimates of the model parameters, resulting in a circularity that makes the problem difficult to solve.

Researchers modeling nonlinear aerodynamics have also used splines in one or two independent variables to fit data with both local[5,6] and global[7,8] models. Klein and Batterson[9] included spline functions in the pool of postulated terms for stepwise regression to identify both local and global nonlinear aerodynamic models. Splines are piecewise polynomials with matched amplitude and derivative conditions at fixed points in the independent variable space, called knots. In general, spline functions are computationally difficult for more than two independent variables and can introduce significant curvature between knots. With the exception of approaches like Ref. 9, modeling with spline functions includes no systematic approach to determining an adequate model structure, so that noise in the data is matched on an equal basis with the real underlying functional dependence. In addition, models that include spline functions make the physical dependence less clear and are generally difficult to update based on new data.

Previous investigations targeted specifically toward modeling the nonlinear aerodynamics of the F-18 High Angle of Attack Research Vehicle (HARV) tabular wind-tunnel data base have resulted in models obtained using ad hoc methods and functions applied at a single Mach number, with some interpolation required.[10,11] The analytical modeling in Refs. 10 and 11 was motivated by the need for smooth derivatives of the aerodynamic coefficients in various control system design and optimization studies. A more common approach to aerodynamic modeling is to use a collection of local models valid at specific flight conditions that encompass the flight envelope of interest.[12,13]

The present work was initiated to investigate the suitability of multivariate orthogonal functions generated from the data for the task of building global nonlinear aerodynamic models with good predictive capability. For the reasons cited above, it was desired that the final form of the model resemble a truncated multivariable power series expansion for the dependent variable in terms of the independent variables. A two-stage technique was developed to achieve an accurate model of the desired form—first, model structure was determined using multivariate orthogonal functions, then the retained multivariate orthogonal functions were decomposed into ordinary polynomial terms in the independent variables. To demonstrate the technique, a global model was identified for $C_Z$ from the wind-tunnel aerodynamic data base for the F-18 HARV.[14] The result was a single compact nonlinear model valid over the entire subsonic flight envelope.

The theory of modeling a function of one independent variable using orthogonal functions generated from the data was developed by Forsythe.[15] Weisfeld[16] extended the orthogonal function generation scheme to functions of several independent variables. These works appeared in the late 1950s. At least one study[17] was done where the theory outlined by Forsythe and Weisfeld was applied to a practical problem—modeling the properties of steam as a function of two independent variables. However, at that time, computer memory was insufficient to solve modeling problems with many independent variables and/or large amounts of data, and the method appears to have been abandoned. With the dramatic increase in computer memory since the late 1950s, along with the techniques developed in this article for model structure determination using orthogonal functions and expressing orthogonal functions in terms of ordinary polynomials in the independent variables, this theory became attractive for modeling nonlinear aerodynamic coefficients. The current work extended and applied the theory described in Refs. 15 and 16 to the problem of modeling nonlinear aerodynamic coefficient functions, including both model structure determination and parameter estimation.

The next section describes the theoretical development. Following this, an example of the nonlinear modeling technique applied to the $C_Z$ of the F-18 HARV is presented, along with some model validation results and a comparison of analytical derivatives to those obtained using finite differences on the tabular wind-tunnel data base.

## Theoretical Development

### Problem Statement

Assume an $N$-dimensional vector of dependent variable values $\boldsymbol{y} = [y_1, y_2, \ldots, y_N]^T$, expressed in terms of a linear combination of $n$ mutually orthogonal functions $\boldsymbol{p}_j$, $j = 1, 2, \ldots, n$. Each $\boldsymbol{p}_j$ is an $N$-dimensional vector, which in general depends on $m$ vectors, $\boldsymbol{x}_i$, $i = 1, 2, \ldots, m$, where each $\boldsymbol{x}_i$ represents an $N \times 1$ vector of the $i$th independent variable. Let $X = [\boldsymbol{x}_i, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m]$, so that $\boldsymbol{p}_j = \boldsymbol{p}_j(X)$. In the following development, the dependence of the $\boldsymbol{p}_j$ on $X$ is understood, but not explicitly shown in the notation. The dependent variable $y$ is then modeled in terms of a linear expansion in the $p_i$

$$y = a_1 p_1 + a_2 p_2 + \cdots + a_n p_n + e \tag{1}$$

where

$$p_i^T p_j = 0 \qquad i \neq j, \quad i, j = 1, 2, \ldots, n \tag{2}$$

The $a_j$, $j = 1, 2, \ldots, n$, are constant model parameters and $e$ denotes the modeling error vector. We put aside for the moment the question of how to determine the orthogonal functions $p_j$ for $j = 1, 2, \ldots, n$, as well as how to select which orthogonal functions should be included in the model of Eq. (1) (which also determines $n$). Now define an $N \times n$ matrix $P$

$$P = [p_1, p_2, \ldots, p_n] \tag{3}$$

and let $a = [a_1, a_2, \ldots, a_n]^T$. Then using (1)

$$y = Pa + e \tag{4}$$

The goal is to determine $a$ so that the least-squares cost function

$$J = (y - Pa)^T (y - Pa) \tag{5}$$

is minimized. The least-squares estimate for $a$ is[15]

$$\hat{a} = (P^T P)^{-1} P^T y \tag{6}$$

**Model Structure Determination**

If the functions in the columns of matrix $P$ were ordinary polynomials in the independent variables with near linear dependence, Eq. (6) would give erroneous estimates $\hat{a}$, due to the ill-conditioning of the $P^T P$ matrix. In addition, an inversion of the $P^T P$ matrix would have to be done for each set of candidate polynomial functions considered for the model (implicitly including various values for $n$). This is done, e.g., in stepwise regression techniques.[1-4] When the functions in $P$ are orthogonal, as assumed here, $P^T P$ is a diagonal matrix, and the problems of inverting an ill-conditioned matrix disappear. In addition, the least-squares problem is then decoupled, since each row of (6) can be solved independently. For the $j$th orthogonal function in $P$, the corresponding row of (6) gives

$$\hat{a}_j = (p_j^T y / p_j^T p_j) \tag{7}$$

The fact that this decoupling does not occur with ordinary polynomials makes model structure determination difficult because the values of all parameters in $a$ depend on all polynomial functions included in the model. Equation (7) shows that when the $p_j$ are orthogonal, each $a_j$ depends only on the measured values of the dependent variable $y$ and the corresponding orthogonal function $p_j$. This means that any term $a_j p_j$ in Eq. (1) is independent of any of the other terms included in the model. Therefore, the selection of the number $n$ of orthogonal functions to be included in the model, as well as which particular $p_j$ should be included as one of those $n$ orthogonal functions, can be done by examining the independent contribution of each orthogonal function to reducing the cost $J$. The expression for the independent cost reduction due to each orthogonal function is developed next.

Substituting the definitions of $a$ and $P$ into Eq. (5), the cost can be expressed as

$$J = y^T y - 2 \sum_{j=1}^{n} a_j p_j^T y + \sum_{j=1}^{n} \sum_{i=1}^{n} a_j a_i p_j^T p_i \tag{8}$$

The assumed orthogonality of the $p_j$ functions [Eq. (2)], along with Eq. (7), transforms Eq. (8) into

$$\hat{J} = y^T y - 2 \sum_{j=1}^{n} \hat{a}_j^2 p_j^T p_j + \sum_{j=1}^{n} \hat{a}_j^2 p_j^T p_j = y^T y - \sum_{j=1}^{n} \hat{a}_j^2 p_j^T p_j \tag{9}$$

where $\hat{J}$ is used in place of $J$ because estimates of $a_j$ were introduced from Eq. (7).

Equation (9) shows that the benefit of including each orthogonal function can be quantified in terms of direct reductions in the squared-fit error, independently of which orthogonal functions are already included in the model of Eq. (1). This makes the model structure determination [choosing the functions in matrix $P$, which implicitly includes determination of $n$ in Eq. (1)] a matter of determining how many functions one is willing to pay for a clearly specified squared-fit error, given by $\hat{J}$. The scalar values $\hat{a}_j^2 p_j^T p_j$ were used to rank the $p_j$ in order of greatest to least contribution to reduction in the squared fit error. With this ranking, $n$ was chosen based on a criterion that represented a tradeoff between reduction in squared fit error and the number of terms required in the final form of the model. The criterion and its use in determining an adequate model structure will be described next.

The mean squared error (MSE) can be defined as

$$\mathrm{MSE} \equiv (1/N)(y - \hat{y})^T (y - \hat{y}) = (1/N)\hat{J} \tag{10}$$

where

$$\hat{y} = P\hat{a} \tag{11}$$

The MSE quantifies the model fit to the data, but does not account for the number of terms in the model. A reduction in MSE could be made by simply adding more terms to the model in Eq. (1). This is because each added orthogonal function term would produce an additional positive term inside the summation of Eq. (9), provided each new orthogonal function is, in fact, orthogonal. Thus, the MSE provides no guard against overfitting the modeling data set. An adequate model must fit the data well, but at the same time have good predictive capabilities. The latter requires a compact model, as discussed previously.

Another quantity, the predicted squared error (PSE), was introduced to account for the number of terms in the model. PSE was defined as[18]

$$\begin{aligned} \mathrm{PSE} &\equiv (1/N)(y - \hat{y})^T (y - \hat{y}) + 2\sigma_p^2 (n_t/N) \\ &= \mathrm{MSE} + 2\sigma_p^2 (n_t/N) \end{aligned} \tag{12}$$

where $\sigma_p^2$ is a constant to be discussed below, and $n_t$ is the number of terms in the model. The first term on the right side of Eq. (12), MSE, measures the fit error. The other term is a linear penalty to prevent including too many terms in the model, and thus overfit the data. This overfit penalty term is intentionally biased high in order to prevent overly complex models and also to account for the fact that the residual vector $(y - \hat{y})$, in general, will not be white (see Ref. 18 for details). The constant $\sigma_p^2$ is the a priori upper bound estimate of the expected squared error between future data and the model, i.e., the upper bound MSE for prediction cases. A simple estimate independent of model structure was used, as recommended in Ref. 18 and described next.

The maximum possible value for $\sigma_p^2$ was considered to be that corresponding to a constant model consisting of the average of the measured values. Denoting this maximum value by $\sigma_0^2$

$$\sigma_0^2 = \frac{1}{N} \sum_{i=1}^{N} (y_i - \bar{y})^2 \tag{13}$$

where

$$\bar{y} = \frac{1}{N}\sum_{i=1}^{N} y_i$$

Assuming $\sigma_p^2$ to be a random variable with uniform distribution, and $0 \le \sigma_p^2 \le \sigma_0^2$, the mean value $(\sigma_0^2/2)$ was used for $\sigma_p^2$. Then substituting for $\sigma_p^2$ in Eq. (12)

$$\text{PSE} = \text{MSE} + \sigma_0^2(n_t/N) \tag{14}$$

with $\sigma_0^2$ defined by Eq. (13).

The PSE from Eq. (14) was evaluated as each orthogonal function was added to the proposed model with the ordering determined by the greatest to least contribution to reduction in the squared fit error. At some point, the reduction in the fit error for additional orthogonal functions added to the model was purchased with too many additional terms in the final form of the model. This point was marked by a minimum value of PSE, which defined an adequate model structure with good predictive capability. With all the $\boldsymbol{p}_j$ mutually orthogonal, each added term necessarily decreases the MSE while increasing the model overfit term in the PSE, thus assuring a clear global minimum for the PSE. More details on the statistical properties of the PSE and justification for its use in modeling problems can be found in Ref. 18.

Using orthogonal functions to model the dependent variable made it possible to order the modeling functions in terms of their respective contributions to reduction in the fit error and to evaluate the merit of including each function individually as part of the model, using PSE. This approach made model structure determination a well-defined and straightforward process.

### Orthogonal Function Generation

The technique for generating orthogonal functions of several independent variables based on the data (the functions $\boldsymbol{p}_j$) will now be outlined. The presentation given here is based on material in Ref. 16. As will be seen, each $\boldsymbol{p}_j$ in general depends on $X$. Let $\boldsymbol{x}_i$, $i = 1, 2, \ldots, m$ again represent $N \times 1$ vectors of the $m$ independent variables. Each element of the $\boldsymbol{x}_i$ corresponds to one data point. Assign $k$ as an ordered positive integer that serves as a label for a unique set of $m$ non-negative integers, $\{k_1, k_2, \ldots, k_m\}$. For example, if $m = 2$, it might be that $k = 1 \Leftrightarrow \{0, 0\}$, $k = 2 \Leftrightarrow \{0, 1\}$, $k = 3 \Leftrightarrow \{1, 0\}$, $k = 4 \Leftrightarrow \{0, 2\}$, $k = 5 \Leftrightarrow \{1, 1\}$, etc. Orthogonal function $\boldsymbol{p}_k$ is associated with the $k$th set of $m$ non-negative integers, and also with the polynomial function $\boldsymbol{w}_k$ of the $m$ independent variables:

$$\boldsymbol{w}_k = \boldsymbol{x}_1^{k_1} \otimes \boldsymbol{x}_2^{k_2} \otimes \cdots \otimes \boldsymbol{x}_m^{k_m} \tag{15}$$

where, e.g., $\boldsymbol{x}_1^{k_1}$ denotes an $N$-dimensional vector with each element of vector $\boldsymbol{x}_1$ raised to the $k_1$ power, and $\otimes$ denotes an element-by-element product. Note that each $\boldsymbol{w}_k$ is an ordinary polynomial in the independent variables. Define the order of $\boldsymbol{w}_k$ as $\varphi(k) = k_1 + k_2 + \cdots + k_m$. Then the orthogonal functions of $m$ independent variables are defined by the following generating relations:

$$\boldsymbol{p}_1 = \boldsymbol{1} \tag{16}$$

where $\boldsymbol{1}$ represents an $N \times 1$ vector of ones, and $k = 1$ is associated with the set of $m$ zeros, $k_1 = k_2 = \cdots = k_{m-1} = k_m = 0$, or $\{0, 0, \ldots, 0\}$. Each new set $k$ evolves from a previous set $\bar{k}$, related as follows:

$$\begin{aligned} \bar{k} &\Leftrightarrow \{k_1, k_2, \ldots, k_{\mu-1}, k_\mu, k_{\mu+1}, \ldots, k_m\} \\ k &\Leftrightarrow \{k_1, k_2, \ldots, k_{\mu-1}, k_\mu + 1, k_{\mu+1}, \ldots, k_m\} \end{aligned} \tag{17}$$

where $\mu$ is an integer. In Eq. (17), the only difference between the integers in set $\bar{k}$ and those in set $k$ is that the integer index for independent variable $\mu$ in set $k$ is one more than in set $\bar{k}$. By assumption, the $\bar{k}$th orthogonal function has already been generated. The orthogonal function $\boldsymbol{p}_k$ is then generated by

$$\boldsymbol{p}_k = \boldsymbol{x}_\mu \otimes \boldsymbol{p}_{\bar{k}} - \sum_j \phi_k^j \boldsymbol{p}_j \tag{18}$$

with the summation over all $j < k$ such that $\varphi(k) - \varphi(j) \le 2$. The $\gamma_k^j$ are constants to be determined. The index $k$ and its associated integer set keep track of the independent variable orders for the $k$th orthogonal function. Each new orthogonal function must be orthogonal only to the previously generated orthogonal functions of the same order, one order lower, and two orders lower to be orthogonal to the entire set of generated orthogonal functions. Proof of this was found by Weisfeld.[16] Orthogonality was verified numerically by the author in the program developed for this work.

The scalars $\gamma_k^j$ are computed for each $j$ by multiplying Eq. (18) by $\boldsymbol{p}_j$ transpose and invoking the orthogonality of the $\boldsymbol{p}_j$, $j = 1, 2, \ldots, k$:

$$\gamma_k^j = \boldsymbol{p}_j^T(\boldsymbol{x}_\mu \otimes \boldsymbol{p}_{\bar{k}})/\boldsymbol{p}_j^T\boldsymbol{p}_j \tag{19}$$

The quantities on the right in Eq. (19) are either a measured independent variable vector or a previously generated orthogonal function. After the scalars $\gamma_k^j$ are determined from Eq. (19), $\boldsymbol{p}_k$ can be computed from Eq. (18). The process can be repeated to generate orthogonal functions of arbitrary order in the independent variables, subject only to limitations related to the information contained in the data.

### Parameter Estimation

Once the orthogonal functions to be included in the model of Eq. (1) were generated by Eqs. (18) and (19), then selected by minimizing the PSE from Eq. (14), each retained orthogonal function was decomposed into an expansion of ordinary polynomial functions in the independent variables. This step introduced little error, as described below.

From Eq. (18) and the discussion in the previous section, it can be deduced that for any index $k$, $\boldsymbol{p}_k$ is a linear combination of the $\boldsymbol{w}_i$ for all $i \le k$. In other words, each $\boldsymbol{p}_k$ can be expressed as a linear combination of the $\boldsymbol{w}_i$, $i = 1, 2, \ldots, k$, which are the ordinary polynomials corresponding to the integer sets of all previously generated orthogonal functions plus the $k$th. In equation form

$$\boldsymbol{p}_k = b_{k1}\boldsymbol{w}_1 + b_{k2}\boldsymbol{w}_2 + \cdots + b_{kk}\boldsymbol{w}_k \tag{20}$$

where the $b_{ki}$, $i = 1, 2, \ldots, k$, are constants to be determined. There is no question as to the model structure given for each $\boldsymbol{p}_k$ in Eq. (20), because this model structure was guaranteed by the use of Eq. (18) in generating each orthogonal function.

As an example, for $m = 2$, let $k = 5$ be the label for the set $\{1, 1\}$. Then the following expansion was used to represent $\boldsymbol{p}_5$ in terms of ordinary polynomial functions:

$$\boldsymbol{p}_5 = b_{51}\boldsymbol{w}_1 + b_{52}\boldsymbol{w}_2 + b_{53}\boldsymbol{w}_3 + b_{54}\boldsymbol{w}_4 + b_{55}\boldsymbol{w}_5 \tag{21}$$

where $\boldsymbol{w}_1$, $\boldsymbol{w}_2$, $\boldsymbol{w}_3$, $\boldsymbol{w}_4$, and $\boldsymbol{w}_5$ are ordinary polynomial functions corresponding to the integer sets for the orthogonal functions $\boldsymbol{p}_1$, $\boldsymbol{p}_2$, $\boldsymbol{p}_3$, $\boldsymbol{p}_4$, and $\boldsymbol{p}_5$, respectively. Since the number of ordinary polynomial terms required for each orthogonal function was known from expansions like Eq. (21), it was possible to use a value for $n_t$ in Eq. (14) for the PSE that corresponded to the number of ordinary polynomial terms in the final form of the model. This point was crucial to determination of a compact adequate model in terms of ordinary polynomials in the independent variables.

The $b_{ki}$, $i = 1, 2, \ldots, k$, in Eq. (20) for the decomposition of each orthogonal function were computed using a conventional least-squares solution, as given in Eq. (6). Since the form and number of terms needed for the ordinary polynomial expansion of each orthogonal function was known, the decomposition of the orthogonal functions into an ordinary polynomial expansion introduced very little error—on the order of the numerical precision ($10^{-12}$)—for each orthogonal function.

When all retained orthogonal functions were decomposed, the expansions like Eq. (20) were substituted into Eq. (1) for each retained orthogonal function, and common terms in the ordinary polynomials were combined using double precision arithmetic to arrive finally at a multivariate model using only ordinary polynomials in the independent variables. Ordinary polynomial coefficients with absolute value less than $10^{-6}$ were dropped from the final model. The program developed for this work uses the procedure described above, and can determine up to eighth-order models with a maximum of 10 independent variables. These limits were imposed by computer memory constraints only.

In summary, a model given by Eq. (1) of the dependent variable in terms of $\boldsymbol{p}_j$ was determined using the minimum PSE criterion. Each $\boldsymbol{p}_j$ in the expansion (1) was then decomposed into an expansion of ordinary polynomial functions, with the results combined to arrive at a model of the form:

$$\boldsymbol{y} = c_1\boldsymbol{w}_{i_1} + c_2\boldsymbol{w}_{i_2} + \cdots + c_{n_p}\boldsymbol{w}_{i_{n_p}} \tag{22}$$

with

$$i_i, i_2, \ldots, i_{n_p} \in \{1, 2, \ldots, n_{\max}\} \tag{23}$$

where $\boldsymbol{w}_j$, $j = i_1, i_2, \ldots, i_{n_p}$ are ordinary polynomials in the independent variables. The positive integer values of $i_1$, $i_2, \ldots, i_{n_p}$ and $n_p$ depend on the particular orthogonal functions retained in the model structure determination stage, and also on the subsequent decomposition of each retained orthogonal function in terms of ordinary polynomials. The number of ordinary polynomial terms $n_p$ may be different than $n$ (from the orthogonal function expansion).

The orthogonal function model of Eq. (1) was useful in determining the model structure for the dependent variable using the minimum PSE criterion, by virtue of the properties of orthogonal functions and the resultant decoupling of the associated least-squares problem. The subsequent decomposition of the retained orthogonal functions was done to express the results in physically meaningful terms and to allow analytical differentiation for derivatives of the dependent variable with respect to the independent variables.

## Example

The $Z$ body axis aerodynamic force coefficient based on the F-18 HARV wind-tunnel data base for a rigid aircraft flying out of ground effect with landing gear retracted, no external stores, and no speed brake deflection, can be represented functionally as

$$C_Z = C_Z(\alpha, M, qc/2V, \delta_s, \delta_n, \delta_f) \tag{24}$$

This example was chosen because it is representative of realistic global nonlinear aerodynamic modeling problems using experimental data.

Unsteady aerodynamic effects, which depend on $\dot{\alpha}$, were combined with the aerodynamics due to $q$. This can be done because $\dot{\alpha}$ and $q$ are nearly the same for most maneuvers. The functional form of the model shown in Eq. (24) results from setting $\dot{\alpha} = q$ for data collection from the wind-tunnel data base, which includes unsteady effects. Therefore, the terms in Eq. (24) associated with $q$ include the effects of both $q$ and $\dot{\alpha}$. In general, when both $q$ and $\dot{\alpha}$ dependencies are included in the model, their similarity leads to identifiability problems when attempting to update the model based on additional data (e.g., from flight tests).

In Ref. 14, the lift and drag coefficients are built up from a set of component functions, with each component function determined by a table lookup in the F-18 HARV wind-tunnel data base. For the present work, these component functions were combined and resolved along the $Z$ body axis to produce data for $C_Z$. The orthogonal function modeling procedure described in this work was used to model $C_Z$ in the following functional form:

$$\begin{aligned} C_Z &= C_{Z_1}(\alpha, M, \delta_s) + C_{Z_2}(\alpha, M, qc/2V) \\ &\quad + C_{Z_{\delta_n}}(\alpha, M)\delta_n + C_{Z_{\delta_f}}(\alpha, M)\delta_f \end{aligned} \tag{25}$$

The four functions in Eq. (25) were modeled individually over the independent variable ranges given below:

$$\begin{gathered} -0.07 \text{ rad} \le \alpha \le 1.57 \text{ rad} \\ (-4 \text{ deg}) \qquad (90 \text{ deg}) \end{gathered} \tag{26}$$

$$0.2 \le M \le 0.9 \tag{27}$$

$$\begin{gathered} -0.524 \text{ rad/s} \le q \le 0.524 \text{ rad/s} \\ (-30 \text{ deg/s}) \qquad (30 \text{ deg/s}) \end{gathered} \tag{28}$$

$$\begin{gathered} -0.419 \text{ rad} \le \delta_s \le 0.183 \text{ rad} \\ (-24 \text{ deg}) \qquad (10.5 \text{ deg}) \end{gathered} \tag{29}$$

$$\begin{gathered} -0.052 \text{ rad} \le \delta_n \le 0.576 \text{ rad} \\ (-3 \text{ deg}) \qquad (33 \text{ deg}) \end{gathered} \tag{30}$$

$$\begin{gathered} -0.140 \text{ rad} \le \delta_f \le 0.785 \text{ rad} \\ (-8 \text{ deg}) \qquad (45 \text{ deg}) \end{gathered} \tag{31}$$

The independent variable ranges in expressions (26–31) represent all the subsonic data available in the F-18 HARV wind-tunnel data base.

Equation (25) was used in lieu of the general form of Eq. (24), because the functional form of Eq. (25) was known from the buildup of aerodynamic force coefficients inside the F-18 HARV simulation, and no nonlinear modeling capability would have been demonstrated by pretending not to know this. Furthermore, decomposing the problem allowed the modeling program to run faster and use less memory. The model also could have been determined directly from the functional form shown in Eq. (24) as a single function of six independent variables.

For the case at hand, the number of independent variables was either two or three. The order of the independent variables ($k_1, k_2, \ldots, k_m$), and the order of the generated multivariate orthogonal functions $\varphi(\boldsymbol{k})$, were limited to the maximum values allowed by program memory constraints [i.e., $k_1 \le 8$, $k_2 \le 8, \ldots, k_m \le 8$, and $\varphi(\boldsymbol{k}) \le 8$]. Outside of the assumption that an eighth-order dependence should be the maximum required, no a priori assumptions were made concerning model structure. The terms for the model were determined automatically by the algorithm described above, using the minimum PSE criterion to select appropriate multivariate orthogonal functions, then decomposing each retained orthogonal function in terms of ordinary polynomials. Orthogonal functions of low order that significantly reduced the MSE were preferred because they involved relatively fewer ordinary polynomial terms and thus minimized the PSE. The complete global analytical model for $C_Z$ required 19 orthogonal function terms and 22 terms in the resulting ordinary polynomial model. The final form of the polynomial model was

$$\begin{aligned} C_Z &= c_1 + c_2\alpha + c_3\alpha^2 + c_4\delta_s + c_5\alpha^3 + c_6\alpha\delta_s + c_7\alpha^2\delta_s \\ &+ c_8M + (c_9 + c_{10}\alpha + c_{11}\alpha^2 + c_{12}\alpha^3 + c_{13}\alpha^4 + c_{14}M \\ &+ c_{15}\alpha M + c_{16}\alpha^2M)(qc/2V) + (c_{17} + c_{18}\alpha)\delta_n \\ &+ (c_{19} + c_{20}\alpha + c_{21}M + c_{22}\alpha^2)\delta_f \end{aligned} \tag{32}$$

Values for $c_i$, $i = 1, 2, \ldots, 22$ are given in Table 1. All angular quantities in Eq. (32) were expressed in radians. The first eight terms in Eq. (32) modeled $C_{Z_1}$, the next eight terms modeled $C_{Z_2}$, and models for $C_{Z_{\delta_n}}$ and $C_{Z_{\delta_f}}$ can be discerned from a comparison of Eqs. (25) and (32).

Figure 1 shows the fit to the $C_{Z_1}$ function over the entire modeled range of $\alpha$ and $\delta_s$, with fixed $M = 0.4$. Experimental values are shown by the wire mesh and the analytical function fit is represented by the smooth surface. Mach number was fixed so that a three-dimensional figure could be drawn. Function fits in other regions of the independent variable space were similar in terms of MSE and the smooth fairing through the experimental wind-tunnel values. The analytical polynomial model faired through a discontinuity in the wind-tunnel data that occurred at $\alpha = 0.698$ rad (40 deg). This discontinuity appeared in the data because two wind-tunnel data sets (one for low $\alpha$, and one for high $\alpha$) were merged to create the F-18 HARV wind-tunnel data base. The global least-squares nature of the analytical model reasonably repaired this local defect, which was clearly unrealistic in its inconsistency with the rest of the data. This demonstrates the robustness of global modeling to local anomalies, which of course includes experimental errors. The robust smoothing by the global analytical model is difficult to see in Fig. 1, but will be apparent in the prediction case described below.

**Table 1 Parameters for a compact global nonlinear model of the F-18 HARV $C_Z$**

| Parameter | Value |
|---|---|
| $c_1$ | 0.86398 E−01 |
| $c_2$ | −0.56828 E+01 |
| $c_3$ | 0.42587 E+01 |
| $c_4$ | −0.78030 E+00 |
| $c_5$ | −0.96090 E+00 |
| $c_6$ | −0.22770 E+00 |
| $c_7$ | 0.44990 E+00 |
| $c_8$ | −0.72250 E−01 |
| $c_9$ | −0.74152 E+01 |
| $c_{10}$ | 0.73036 E+01 |
| $c_{11}$ | −0.40268 E+02 |
| $c_{12}$ | 0.51855 E+02 |
| $c_{13}$ | −0.17351 E+02 |
| $c_{14}$ | −0.52958 E+00 |
| $c_{15}$ | 0.17775 E+02 |
| $c_{16}$ | −0.11967 E+02 |
| $c_{17}$ | 0.15135 E+00 |
| $c_{18}$ | −0.48175 E+00 |
| $c_{19}$ | −0.81942 E+00 |
| $c_{20}$ | 0.57681 E+00 |
| $c_{21}$ | −0.27117 E+00 |
| $c_{22}$ | 0.74850 E+00 |

Figure 2 shows the result of using tabular wind-tunnel aerodynamics (solid lines) vs using the polynomial model given by Eq. (32) (dotted lines) for the same independent variable inputs, shown in Fig. 3. The time histories of the independent variables in Fig. 3 were taken from an actual flight test maneuver. The last three plots of Fig. 3 show the control surface inputs for this longitudinal maneuver, a push-over pull-up initiated at $\alpha = 30$ deg. This maneuver was selected because it covered a wide range of $\alpha$, $M$, $\delta_s$, and $C_Z$. The result in Fig. 2 represents a prediction case for the global analytical model, since the time histories in Fig. 3 were not used to identify the model. The average discrepancy in Fig. 2 was 3.02% of the wind-tunnel value. The match achieved in Fig. 2 is very good, with the polynomial model having the advantages of simplicity, compactness, global validity, analytical differentiability, and improved insight into the character of the nonlinear aerodynamic force.

Most of the discrepancy in the $C_Z$ prediction of Fig. 2 was traced to the smoothing of data discontinuities and extrapolation beyond the limits of the wind-tunnel data base. This is apparent in Fig. 4, which shows time histories of the $C_{Z_1}$, $C_{Z_2}$, $C_{Z_{\delta_n}}$ and $C_{Z_{\delta_f}}$ functions for the same run. In particular, when $\alpha$ passed through 40 deg (at approximately 13.5 and 23.5), significant discontinuities occurred in $C_{Z_1}$, $C_{Z_{\delta_n}}$ and $C_{Z_{\delta_f}}$ computed from the wind-tunnel data base, as a result of the aforementioned merging of two wind-tunnel data sets. The global analytical model made reasonable reparations of these local discontinuities, as can be seen in Fig. 4.

The wind-tunnel data base values for $C_{Z_{\delta_n}}$ and $C_{Z_{\delta_f}}$ were zero when $\alpha$ exceeded 40 deg, apparently because no data existed for these functions at $\alpha > 40$ deg. These zero values were not used in the analytical modeling, so that the variations shown in Fig. 4 for the analytical model were pure extrapolations. This approach was considered more physically plausible than setting function values to zero when $\alpha$ exceeded 40 deg.

The small discrepancy for the $C_{Z_2}$ function in the second plot of Fig. 4 was partly due to the $\dot{\alpha} = q$ assumption in the analytical polynomial modeling, which was discussed previously. In addition, the wind-tunnel data for both $C_{Z_2}$ and $C_{Z_{\delta_n}}$ exhibited relatively large variability in some regions of the independent variable space. The PSE criterion prevents precise modeling of this variability because of the attendant requirement for many more terms in the model and the detri-

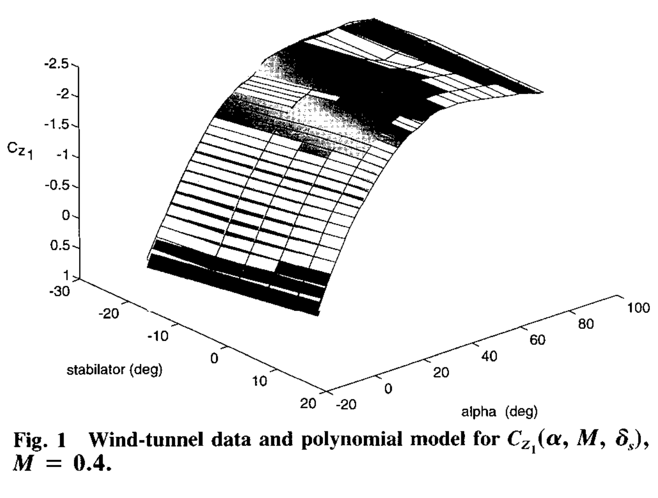


**Fig. 1 Wind-tunnel data and polynomial model for $C_{Z_1}(\alpha, M, \delta_s)$, $M = 0.4$.**

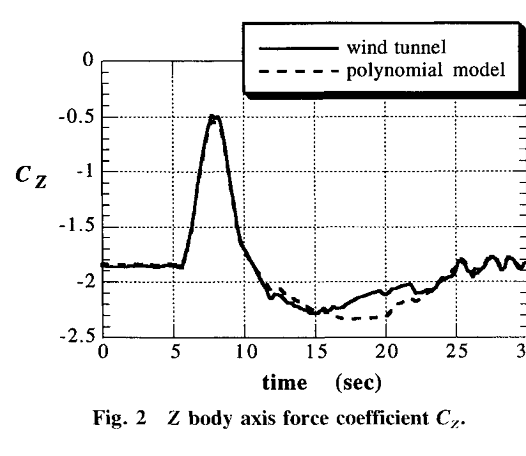


**Fig. 2 Z body axis force coefficient $C_Z$.**

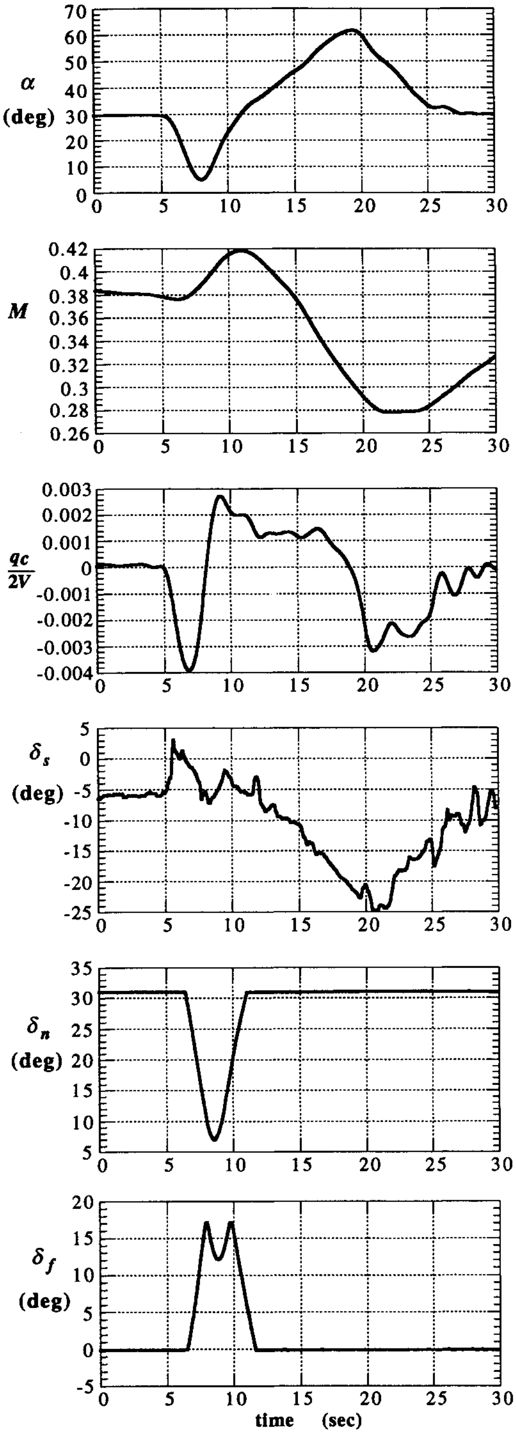


**Fig. 3 Independent variable time histories from flight test.**

**Table 2 Comparison of PSE with MSE for a prediction case**

| Function | PSE from the model determination | MSE for the prediction case (Fig. 4) |
|---|---|---|
| $C_{Z_1}$ | 0.0107 | 0.0032 |
| $C_{Z_2}$ | 4.45 E−05 | 3.87 E−06 |
| $C_{Z_{\delta n}}$ | 0.0089 | 0.0017 |
| $C_{Z_{\delta f}}$ | 0.0141 | 7.69 E−04 |

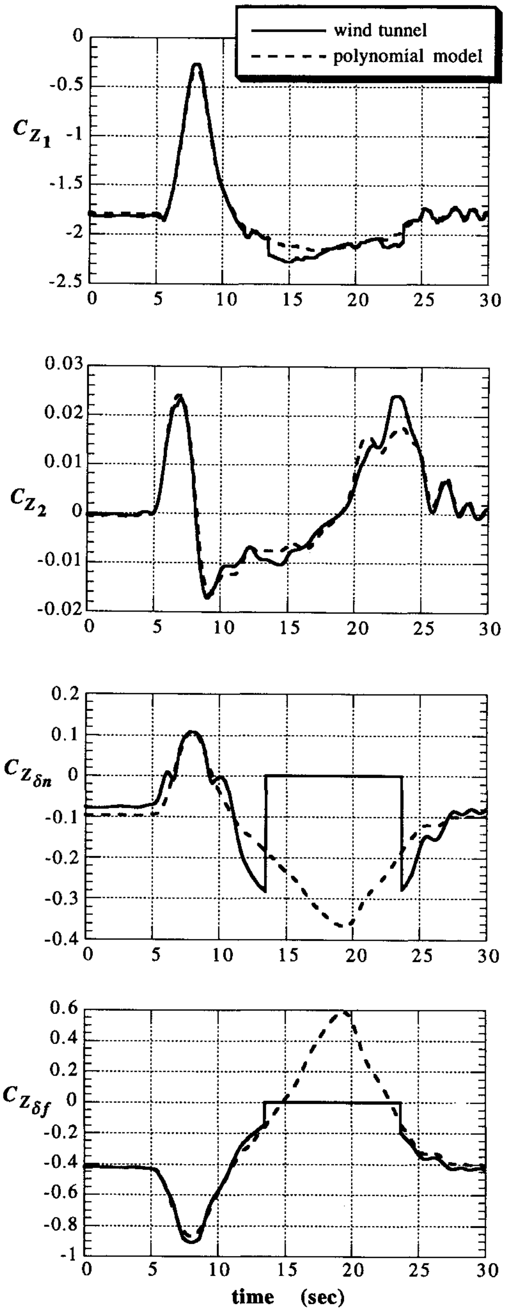


**Fig. 4 $C_Z$ component functions.**

mental effects this would have on predicting the dependent variable in other regions of the independent variable space.

Another (probably small) source of discrepancy in Fig. 4 could be that the values from the wind-tunnel data base were linearly interpolated between tabulated values. It is evident from Figs. 2 and 4 that the small mismatch in modeling $C_Z$ for this prediction case was primarily from smoothing the local discontinuities in the wind-tunnel data and extrapolation for $\alpha > 40$ deg, both of which were considered acceptable and desirable.

In Table 2, PSE computed during the modeling of each function in Eq. (25) was compared to MSE for the prediction case of Fig. 4. The data plotted in Fig. 4 was used in Eq. (10) to compute the MSE for the prediction case. Calculation of MSE for the $C_{Z_{\delta_n}}$ and $C_{Z_{\delta_f}}$ time histories excluded the portion where the analytical model extrapolated ($\alpha > 40$ deg). The values in Table 2 indicate that the PSE gave conservative estimates of MSE for the prediction case. This was expected because the PSE from Eq. (14) is intentionally biased, as explained previously. In this work, the PSE was not used as a quantitative estimator of the MSE for prediction cases, but rather as a tool for model structure determination.

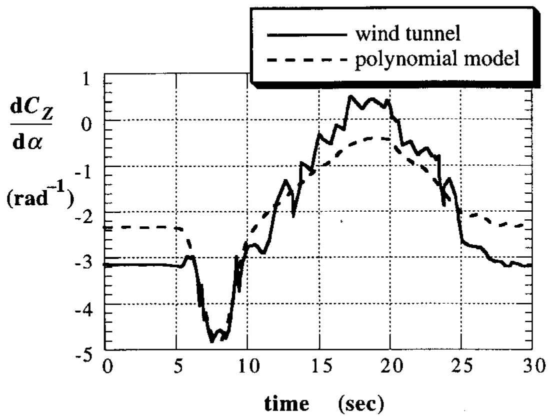


**Fig. 5 Calculated alpha derivative $dC_Z/d\alpha$.**

The form of the analytical model in Eq. (32) was particularly easy to differentiate with respect to the independent variables. The program developed for this work was designed to compute derivatives analytically by automation of the power rule of differentiation from elementary calculus. Figure 5 shows time histories of ($dC_Z/d\alpha$) ($\alpha$ in radians), computed for the same prediction case. The solid line was computed by central finite differences ($\Delta\alpha = \pm 1.0$ deg) using the tabular wind-tunnel data base, and the dotted line represents an analytical derivative of Eq. (32). Figure 5 shows that the local slopes from the global analytical model do not always match closely with local slopes from the wind-tunnel data base. This is because the analytical model is global, and therefore, (roughly speaking) attempts to model only the most prominent features in the data. The analytical derivative is much smoother and may in fact be more accurate, since the derivative is exact and the global analytical model smoothes the local discontinuities and experimental error that degrade the accuracy of the finite difference derivative.

## Concluding Remarks

A new technique for modeling nonlinear aerodynamic coefficients using multivariate orthogonal functions was developed. The technique used a minimum predicted squared-error criterion and the orthogonality of the modeling functions to determine an adequate model structure by decoupling the associated least-squares problem. After the model structure was determined using multivariate orthogonal functions, each orthogonal function was broken down into ordinary polynomials in the independent variables. This latter step was straightforward because the structure of each generated multivariate orthogonal function was known from the generation process. After collecting like terms, the final model could be interpreted as selectively retained terms from a multivariable power series. The method is general, although it was developed for the purpose of modeling nonlinear aerodynamic coefficients.

The approach was described in detail, then demonstrated by constructing a global analytical model for $C_Z$ of the F-18 HARV based on wind-tunnel data. For a realistic prediction case, $C_Z$ was modeled within 5% of the wind-tunnel values using the global polynomial model. In addition, the global polynomial model exhibited desirable characteristics in terms of robustness to local data anomalies and reasonable extrapolation properties.

The modeling technique described here was shown to be capable of generating a compact, global analytical representation of nonlinear aerodynamics with good predictive capabilities from which smooth analytical derivatives of any order can be determined easily. In addition, the technique provides insight by identifying the independent variable combinations that are most important in the global characterization of the nonlinear aerodynamics.

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